SPECTRAL ASPECTS OF AVERAGE VERTEX CONNECTIVITY OF A GRAPH

MSc. (MATHEMATICAL SCIENCES)

THESIS

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UNIVERSITY OF MALAWI

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DECLARATION

I, the undersigned, hereby declare that this thesis is my own original work which has not been submitted to any other institution for similar purposes. Where other people's work has been used, acknowledgments have been made.

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CERTIFICATE OF APPROVAL

The undersigned certify that this thesis represents the student's own work and effort and has been submitted with our approval.

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ABSTRACT

The measures of global connectivity such as graph integrity and toughness have non-polynomial time complexities. This has led to the development of global average graph vertex connectivity measures that are dependent on combinatoric counting of number of internally disjoint paths in a graph. Many results related to average vertex connectivity have been found. This study develops a spectral form of average vertex connectivity, together with its upper bounds. Using trees, we demonstrate that the new definition and its upper bounds are more related to ordinary graph parameters.

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ABBREVIATIONS AND ACRONYMS

A	Adjacency matrix of a graph
L	Laplacian matrix
K_G	Connectivity of a graph G
$\overline{K}_G$	Average connectivity of graph G
$(A^l)_{uv}$	Number of u-v walks of length l
d	Diameter of a graph
r	Radius of a graph
M_G	Connectivity of a graph in spectral form
$\overline{M}_G$	Spectral average vertex connectivity of a graph
NP-hard	Non-deterministic polynomial-time hardness computationally
n	Order of a graph
D	Diagonal matrix

CHAPTER 1

INTRODUCTION

1.1. Introduction and background

One most important tool mathematicians have developed is the application of linear algebra to Graph Theory. This has become to almost a stand alone field of Algebraic Graph Theory that uses spectral properties of adjacency matrix, A, of a graph.

It has been shown that $(A)^l_{uv}$ is the number of walks of length l (Godsil & Royle, 2001). The question to ask is whether it is possible to count the number of internally disjoint paths k_{uv} in a graph say G. The problem one can have is that of counting paths from walks, and it is very reasonable to assume that a graph has fewer $uv-$ paths than $uv-$ walks, and also significant to assume that a graph has fewer paths that are internally disjoint. This is a very interesting puzzle. Since $(A)^l_{uu}$ is the number of closed walks in a graph G with adjacency matrix A, then trace(A^l) must guide a mathematician in a process of finding the total number of closed walks and hence simplifying the task of calculating the average vertex connectivity of a graph.

Connectivity strength can be deduced using a formula that takes number of vertices, n, into account. Beineke et al. (2002) developed independently two similar formulas for that using combinatoric techniques. They have done so by dividing the total number of internally disjoint paths in a graph by $\frac{n(n-1)}{2}$ or $n(n-1)$. These quantities are called *average vertex connectivity* and are denoted by $\overline{k}_G$. That is $\overline{k}_G = \dfrac{\sum_{u,v} k(u,v)}{\binom{n}{2}}$ (Bermond et al., 1984) or $\overline{k}_G = \frac{\sum_{u,v} k(u,v)}{n(n-1)}$ (Bollobas, 2011) where $k(u,v)$ is the number of internally disjoint paths between u and v. The authors also observed that the average connectivity $\overline{k}$ is bounded above by $n-1$ where the graph is complete. Clearly, the total number of internally disjoint paths has no closed form in terms of order n of a graph, otherwise this lemma (which states that if G has order n, then $\overline{k}_G \leqslant n-1$, with equality if and only if G is complete) (Beineke et al., 2002) would talk of exactness and not upper bound.

The authors determined upper bounds of average vertex connectivity and gave combinatoric proofs. It is clearly seen in Beineke et al. (2002) that average connectivity gives strength of connectivity that is similar to integrity and toughness though it is easier to compute in polynomial time than the latter. However, the measure of connectivity in Beineke et al. (2002) is not applicable to all families of graphs, like in a tree.

One way of improving Beineke et al. (2002) work is to come up with spectral approximation to the total number of internally disjoint paths. This would also assist in relating well the connectivity to what happens in a graph especially the valence sequence. That approach has been taken in this study, and literature does not have similar work. We use this definition to give the upper bounds of average connectivity. Since information of a graph is easily stored in the adjacency matrix A, using utile properties of adjacency matrix to estimate the total number of internally disjoint paths would be much easier than simply counting.

Connectivity has a substantial real life application in disciplines such as Computer networks and Electricity (determination of positioning of redistribution points or communication end points that can affect so many other points when disrupted), Epidemiology (to help in disrupting the contact networks to prevent propagation of an infection), and in Criminology (to help in interrupting the criminal networks created in societies).

1.2. Research Objectives

1.2.1 Main Objective

The main objective of the research is to propose a spectral representation of average vertex connectivity and its upper bounds.

1.2.2 Specific Objectives

Specifically, we would like to:

1. give a definition of average vertex connectivity in spectral form.
2. derive upper bounds of average vertex connectivity using adjacency matrix properties.
3. compare tightness of spectral and combinatorial upper bounds of average connectivity.

1.3. Rationale of the Study

Spectral Graph Theory is a well established branch of graph theory that also can be applied in graph theory problems. If successfully applied to approximate average number of internally disjoint paths, the average vertex connectivity would be much easier to calculate. This is so because information of complicated graphs is easily stored in adjacency matrices as compared to drawings in combinatoric graph theory.

1.4. Overview of the thesis

The main contribution of the study is the formulation of the spectral definition of the average vertex connectivity of a graph and its upper bounds. In Chapter 2, we define some basic concepts that are salient to the content of study including some characteristics of a graph, some classes of graphs and properties of an adjacency matrix. In Chapter 3, relevant literature has been reviewed on the concept of connectivity and the global measures of connectivity such as toughness and integrity which led to the development of other global parameters such as average vertex connectivity that can be easily computed in polynomial time. In Chapter 4, methodology and results are presented which include the definition of spectral average vertex connectivity and its justification. It has been shown that the parameter $\overline{M}_G$ gives the absolute bounds in its application in complete graphs, thus more tighter than $\overline{k}_G$. Further, we develop several upper bounds of this spectral average connectivity whose applications are made in a number of families of graphs to show their consistency with average vertex connectivity. Examples are given for clarity. Finally, Chapter 5 presents the conclusion of all the work in the study.

CHAPTER 2

PRELIMINARIES

We present the needed preliminaries relating to graph theory, distance and connectivity of a graph, and fundamental facts in spectral graph theory pertinent to this study, which include properties of adjacency matrix.

2.1. Graph and its characteristics

Definition 1 *Graph (Diestel, 2016)*

A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v; the vertices u and v are called the ends of e.
The order of graph G, written $n(G)$, is the number of vertices in G, and the size of graph G, written $e(G)$, is the number of edges in G.

Definition 2 *Finite, trivial and simple graphs (Bapat, 2010)*

A graph is *finite* if both its vertex set and edge set are finite.

We call a graph with just one vertex *trivial* and all other graphs nontrivial.

A graph is *simple* if it has no loops and no two of its links join the same pair of vertices.

In this study, we focus only on finite graphs, and so the term 'graph' always means 'finite graph'.

Definition 3 *Vertex degrees (Barriere, 2013)*

The degree or valency $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively, of vertices of G. If every vertex of G has the same degree, the graph is called *Regular*. A regular graph with vertices of degree k is called a *k-regular* graph or regular graph of degree k.

We also encounter counting problems about subgraphs. One of which is the counting of edges. This is done by using vertex degrees. The resulting formula is an essential tool of graph theory, sometimes called 'The First Theorem of Graph Theory' or the 'Handshaking Lemma'.

Proposition 1 *Degree-Sum Formula (Plesnik, 1984)*

If G is a graph, then $\sum_{v \in V(G)} d_G(v) = 2e(G)$.

Proof

Summing the degrees counts each edge twice, since each edge has two ends and contributes to the degree at each end point ■

Definition 3 and Proposition 1 have several immediate corollaries such as those given below.

Corollary 1 *(Roberts & Tesman, 2009)*

In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence $\delta(G) \leqslant \frac{2e(G)}{n(G)} \leqslant \Delta(G)$.

Corollary 2 *(Brandt & Veldman, 2009)*

Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree.

Corollary 3 *(Gross et al., 2019)*

A $k-$regular graph with n vertices has $\frac{nk}{2}$ edges.

2.2. Some classes of graphs

Definition 4 *Complete, Empty and Bipartite graphs (Bapat, 2010)*

A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. It is denoted by K_n.

An empty graph, on the other hand, is one with no edges.

A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

Definition 5 *Walks, Trails, Paths and Cycles (Fiol et al., 2014)*

A *walk* in a graph is a sequence of (not necessarily distinct) vertices $v_1, v_2, ..., v_k$ such that $v_i v_{i+1}$ in $E(G)$ for $i = 1, 2, ..., k-1$. Such a walk is sometimes called a $v_1 \breve{} v_k$ walk, and v_1 and v_k are the end vertices of the walk. If the edges $e_1, e_2, ..., e_k$ of a walk are distinct then the walk is called a *trail.*

If the vertices in a walk are distinct, then the walk is a *path.* A family of paths in G is said to be *internally-disjoint* if no vertex of G is an internal vertex of more than one path of the family.

A walk is closed if it has positive length and its origin and terminus are the same. A closed trail whose origin and internal vertices are distinct is a *Cycle.* Just as with paths we sometimes use the term 'cycle' to denote a graph' corresponding to a cycle. A cycle of length k is called a $k-$cycle; a $k-$cycle is odd or even according as k is odd or even. A $3-$cycle is often called a triangle. These lead us to a lemma about walks and paths.

Lemma 1 *(Ghosh & Boyd, 2006)*

In a graph G with vertices u and v, every $u \breve{} v$ walk contains a $u \breve{} v$ path.

proof

Let W be a $u \breve{} v$ walk in G. We prove this theorem by induction on the length of W. If W is of length 1 or 2, then it is easy to see that W must be a path.

For the induction hypothesis, suppose the result is true for all walks of length less than k, and suppose W has length k. Say that W is $u = w_0, w_1, w_2, ..., w_{k-1}, w_k = v$ where the vertices are not necessarily distinct. If the vertices are in fact distinct, then W itself is the desired $u \breve{} v$ path. If not, then let j be the smallest integer such that $w_j = w_r$ for some $r > j$. Let W_1 be the walk $u = w_0, ..., w_j, w_{r+1}, ..., w_k = v$. This walk has length strictly less than k, and therefore the induction hypothesis implies that W_1 contains a $u-v$ path. This means that W contains a $u-v$ path. ■

Lemma 1 helps us to understand graph connectedness as described in the definitions given below.

Definition 6 *(Halin, 1969)*

A graph G is *connected* if it has a $u-v$ path whenever $u, v \in V(G)$ (otherwise, G is *disconnected*). If G has $u-v$ path, then u is *connected* to v in G. The *connection relation* on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.

Definition 7 *(Bauer et al., 2006)*

An graph $G = (V; E)$ is said to be k - connected if $|G| > k$ and we cannot obtain a non-connected graph by removing $k-1$ vertices from V.

Hence, the *vertex-connectivity* or simply *connectivity* k_G of a graph G is the minimum cardinality of a vertex-cut (set of all vertices distinct from two nonadjacent vertices) of G if G is not complete, and $k_G = n-1$ if $G = K_n$ for some positive integer n. Hence $k(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Definition 8 *Trees (Beezer, 2009)*

A graph with no cycle is *acyclic*. A *forest* is an acyclic graph. A tree is a connected acyclic graph. A *leaf* (or *pendant vertex*) is a vertex of degree 1. A *spanning subgraph* of G is a subgraph with vertex set $V(G)$. A *spanning tree* is a spanning subgraph that is a tree. These facts lead us to a corollary.

Corollary 4 *(Beezer, 2015)*

(a) Every edge of a tree is a cut-edge.

(b) Adding one edge to a tree forms exactly one cycle.

(c) Every connected graph contains a spanning tree.

2.3. Diameter and Radius

Two of the most commonly observed parameters of a graph are its radius and diameter. The diameter of a connected graph G, denoted $diam(G)$, is the maximum distance between two vertices. The eccentricity of a vertex is the maximum distance from it to any other vertex. The radius, denoted $rad(G)$, is the minimum eccentricity among all vertices of G. Of course the diameter is the maximum eccentricity among all vertices.

For a connected graph $G : rad(G) \leq diam(G) \leq 2rad(G)$ (Bollobas, 1998). The upper bound follows from the triangle inequality. The radius and diameter are easily computed for simple graphs, and the following are some facts about them.

1. Complete graphs: $diam(K_n) = rad(K_n) = 1$ (for $n \geq 2$) (Caccetta & Smyth, 1992).

2. Complete bipartite graphs: $diam(K_{m,n}) = rad(K_{m,n}) = 2$ (if n or m is at least 2).

3. Path on n vertices: $diam(P_n) = n - 1; rad(P_n) = \lceil \frac{n-1}{2} \rceil$.

4. Cycle on n vertices: $diam(C_n) = rad(C_n) = \lfloor \frac{n}{2} \rfloor$. (Dankelman & Oeller-

mann, 2003)

Note that cycles and complete graphs are vertex-transitive, so the radius and diameter are automatically the same (every vertex has the same eccentricity). The centre is the subgraph induced by the set of vertices of minimum eccentricity. Graphs G where $rad(G) = diam(G)$ are called self-centred. The centre of a graph forms a connected subgraph, and is contained inside a block of the graph (West, 2001).

2.4. Adjacency Matrix

We first define an adjacency matrix and thereafter describe some of its properties useful in this study.

Definition 9 *Adjacency Matrix (Sheffer, 2003)*

Let G be a graph with $V(G) = 1, ..., n$ and $E(G) = e_1, ..., e_m$. The adjacency matrix of G, denoted by A_G, is the $n \times n$ matrix whose rows and the columns are indexed by $V(G)$. If $i \neq j$ then the (i, j)-entry of A_G is 0 for vertices i and j non-adjacent, and the (i, j)-entry is 1 for i and j adjacent. The (i, i)-entry of A_G is 0 for $i = 1, ..., n$. We often denote A_G simply by A.

2.4.1 Spectrum of Adjacency Matrix

Studying graph theory using the properties of adjacency matrix A is called *Spectral graph theory.* A spectral property is a property that is related to the spectrum of a matrix, which is an array consisting of numbers called *eigenvalues* and their frequencies are called *multiplicities.*

An eigenvalue of A is a number, λ, such that there exists a non-zero vector, $\overrightarrow{v}$ for which $A\overrightarrow{v} = \lambda\overrightarrow{v}$. Since determination of eigenvalues is hard, we usually find a number, λ, for which the matrix $A - \lambda I$ is singular. Thus I is the matrix such that $AI = IA = A$. If one expresses $A - \lambda I$ in reduced echelon form, the number of rows containing all zeros in matrix $A - \lambda I$ is called the *nullity* of A. The *nullity* represents multiplicity of eigenvalue λ.

Definition 10 *Spectrum of a graph (Sheffer, 2003)*

The spectrum of a graph G is the set of eigenvalues (with multiplicity) of the adjacency matrix of G.

For a graph $G = (V; E)$ with $n = |V|$ vertices, the $n \times n$ adjacency matrix A has n eigenvalues, when counted with multiplicity. Further, since A is symmetric, all n eigenvalues will be real. For small graphs, the easiest way to find the spectrum is to find the roots of the characteristic polynomial $\chi(x) = det(xI - A)$.

Definition 11 *Cospectral graphs (Kim, 2016)*

Graphs with the same spectrum (that is, with adjacency matrices having the same eigenvalues with the same multiplicities) are called *cospectral*.

Definition 12 *Trace*

Let A be an n by n matrix, then the trace of matrix A, $tr(A)$ is defined as $trace(A) = \sum_{k=1}^{n}(A)_{kk}$ (Strang, 2006). This definition leads us to the fundamental lemma.

Lemma 2 *Alternative Definition of Trace of AB (Lipschutz, 1991)*

If A and B are square matrices then $Trace(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} (A)_{ik}(B)_{ki}$.

proof

From the definition of trace, we must have $trace(AB) = \sum_{i=1}^{n} (AB)_{ii}$

However, the definition of elements of a matrix states that $(AB)_{ii} = \sum_{k=1}^{n} (A)_{ik}(B)_{ki}$, and making this substitution into the above equation we have $trace(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} (A)_{ik}(B)_{ki}$ ■

Theorem 2.4.1 *Number of closed walks of a given length (Mader, 1979)*

Consider a graph G with spectrum $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and write $CW(k)$ for the number of closed walks of length k in G. Then $CW(k) = \sum_{i=1}^{n} \lambda_i^k$.

proof

Since the closed walks are precisely the walks that start and end in the same place, we have $CW(k) = \sum_{i \in V} W_{ii}(k)$ and $CW(k) = \sum_{i \in V} (A^k)_{ii} = Trace(A^k)$, the trace of A^k.

We know that the trace of a matrix is the sum of its eigenvalues (Ghosh & Boyd, 2006). Further, we know that the eigenvalues of a matrix power are the powers of the eigenvalues of the original matrix (Ghosh & Boyd, 2006). Hence the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$ and we have $CW(k) = Trace(A^k) = \sum_{i=1}^{n} \lambda_i^n$ ■

CHAPTER 3

LITERATURE REVIEW

3.1. Introduction

In this chapter we look at efforts made by mathematicians in developing means of measuring connectivity and further advances taken to derive methods of global measures of connectivity.

3.2. Classical Connectivity

Since the development of the branch of Mathematics of graph theory, researchers have tried to measure connectivity of graphs (Fiol et al., 2014). The simplest one has been that of just counting the number of edges or vertices needed to be removed in order to create a graph with many components or that which is trivial. Therefore, *connectivity* is the minimum number of edges or vertices to be deleted until we get a trivial or a disconnected graph, usually denoted by symbol K_G.

This connectivity definition has been important in that it has been used by almost all researchers in graph connectivity in order to derive useful properties. A good example is that of Beineke et al.(2002) who use this to determine a lower bound of their newly constructed average connectivity.

Another example is Fiedler (1973), who vigorously uses this definition to show that the second smallest eigenvalue of a Laplacian matrix can measure the algebraic connectivity as well as the strength of connectivity of a graph. That is the second smallest eigenvalue of a Laplacian matrix is *zero* if and only if the graph is disconnected.

The main set back of the definition is that it fails to give the strength of connectivity of a graph (Aslan, 2014). For example, every tree has connectivity of size 1, but some trees have stronger connectivity (Chung, 1988).

3.3. Strength of Connectivity

If a graph represents a network, one may wish to study the number of edges from each point to find out whether they are related to eigenvalues of A or not. The number of edges from a single vertex is called *valency* or *degree* of the vertex.

Usually in networks, someone might be interested in finding the strength of connectivity. The simplest way is that of finding the number of vertices or edges that can be deleted for one to obtain de-linked graphs, which are called *components* . Sometimes this is not possible, so one can keep deleting vertices or edges until there are no edges.

In this case one might believe that the higher the connectivity the higher the strength of the connectivity. However, this has been refuted by researchers studying mainly the matrix $D-A$, where D is a diagonal matrix that contains a sequence of degrees of vertices of a graph in the major diagonal. The second smallest eigenvalue of this matrix, $D - A$, which is usually called *Laplacian matrix*, L, is called *Fiedler Value* or *Algebraic Connectivity.* If this number increases , then it can be hard to disconnect a graph, say G. This measure does not use directly the number of vertices in graph G. In other words, it does not use the order of G, which is contained in A. The number of edges is also called the *size* of G and is also contained in A.

3.4. Average Vertex Connectivity

One natural question could be whether one can deduce the strength of connectivity using a formula that takes number of vertices, n, into account. This question was answered by Beineke et al. (2002) who developed independently two similar formulas for that. The technique was that of considering paths (a path is a trail in which all vertices, except possibly the first and last, are distinct) in a graph that do not share common vertices. Such paths are called *internally disjoint paths.* So one can count these in a graph. If there are no closed paths in a graph then it is called a *tree* if it is connected, otherwise is called a *forest.*

One fact to consider is whether there is a path between every pair of vertices. If such is a case, the graph is called *complete graph.* Any complete graph on n vertices has $\frac{n(n-1)}{2}$ edges.

One simple way of measuring connectivity is by dividing the total number of internally disjoint paths in a graph by $\frac{n(n-1)}{2}$ or $n(n-1)$. These quantities are called *average vertex connectivity* and are denoted by $\overline{k}$. That is $\overline{k}_G = \dfrac{\sum_{u,v} k(u,v)}{\binom{n}{2}}$ (Beineke et al., 2002) or $\overline{k}_G = \frac{\sum_{u,v} k(u,v)}{n(n-1)}$ (Kim, 2016) where $k(u,v)$ is the number of internally disjoint paths between u and v.

3.5. Attempts to Measure Connectivity Strength

Some researchers like Bagga et al. (1993) could believe that if many edges or vertices are removed from a graph to disconnect or trivialize it, then that graph is strongly connected. As such, they created classification schemes for measuring reliability and vulnerability of connectivity in networks. On the contrary, if a graph is a tree, there is only one vertex that can be removed to disconnect it (Abiad et al., 2018). However, there are trees which can have strong connectivity but the vertex that can be removed has higher valence (Dankelmn & Oellermn, 2003). Hence, Bagga et al.(1993) concluded that this is not a global measure of connectivity.

This has led to many researchers to think of global measures of connectivity. One of them is Chvatal (1973) who developed a measure of global connectivity called Toughness. This measure relates the number of components and vertices to be removed for the graph to be disconnected. This measure is important in that its magnitude could give the strength of connectivity, not only as number of vertices to be removed, but also shows the consequences of removing the vertices in the graph, thus, the number of components that can be created upon deletion of vertices. This measure is not easy to understand (Johnson, 1974) and is non-polynomial hard to compute(NP-hard) (Mader, 1979). Hence, this measure has led to creation of many conjectures and theorems related to it. Bauer et al. (2006) listed 99 theorems and conjectures related to toughness.

This problem made Fiedler (1973) consider measuring connectivity using second smallest eigenvalue of Laplacian matrix, as described in the first paragraph. The advantage is that the higher the algebraic connectivity the higher the toughness of the graph (Goddard & Oellerman, 2011). In addition, the measure has been found to be more useful than toughness in analyzing synchronizability and robustness of networks.

Fiedler (1973) proves the property that helps researchers to consider *Fiedler Value* as a measure of global connectivity (West, 2001). The proof considers existence of a *zero eigenvalue* in a Laplacian matrix (Harris et al., 2008). However, it is well known that the lowest eigenvalue of any Laplacian matrix is *zero* (Godsil & Royle, 2001). Nevertheless the *Fiedlers Value* does not relate well to other parameters of a graph, like *valence sequence* (Gross et al., 2019).

Hence, some researchers considered modifying toughness of a graph to develop a better measure. Such attempts led to the development of a measure called Integrity (Chvatal, 1973). Integrity is an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices, and the order of the largest remaining component (Bauer et al., 2006). Clearly, the measure is not directly related to the graph being disconnected, rather it is a parameter of a set of vertices.

3.6. Dealing with NP-hard Connectivity Measures

Since integrity and toughness have non-polynomial hard computation time complexity, just like toughness (Bauer et al., 2006), most researchers considered creating a measure of connectivity that does not only indicate whether a graph is connected or not, but that shows what happens in a graph (Chartrand & Oellerman, 1993). This prompted researchers to investigate working out average vertex or edge connectivity. Focus was put on coming up with a measure that is also computationally efficient.

One way of doing this is that of counting the sum of numbers of internally disjoint paths from each vertex in a graph and dividing it by a number related to vertex count when a simple complete graph is reproduced from the same graph. The first attempt to do this was made by Beineke et al. (2002) who chose to divide the sum of number of internally disjoint paths by twice the order of a complete graph that can be reproduced. The upper bounds of the measure of connectedness so created were derived and established. It is much easier to use network flow techniques in her definition of a measure of connectivity to compute the value of average connectivity so defined. Thus, there was an improvement in computational time.

However, the definition of average connectivity derived by Beineke et al. (2002) could work efficiently only for multipartite graphs. The justification of dividing the sum of the number of internally disjoint paths by twice the order of a simple complete graph that can be reproduced was not justified. A simple improvement of not doubling the order of complete graph that can be reproduced could be suggested. This was adopted by Beineke et al. (2002).

Beineke et al.(2002) definition of average connectivity can be extended to any type of graph. Furthermore, Beineke et al. (2002) observe that the lower bound of their average connectivity is 1, despite the fact that a closed form of number of internally disjoint paths in a tree does not exist.

In addition, Beineke et al. (2002) consider their connectivity to be bounded above by $n - 1$ for a complete graph whose order is n. However, the sum of the upper bound for internally disjoint paths in a simple complete graph is not established.

CHAPTER 4

METHODOLOGY AND RESULTS

4.1. METHODOLOGY

4.1.1 Introduction

We define a measure of global connectivity using algebraic graph theory parameters, especially adjacency matrix, A. We then show that the measure is an upper bound of Beineke et al.(2002) measure of global connectivity in order to justify as why it works by finding an approximation to the sum of numbers of internally disjoint paths using the spectral number of walks lemma.

4.1.2 Spectral Definition of Average Connectivity of a Graph.

Our main contribution is the definition of average vertex connectivity in spectral graph theory. We present this in the following definition.

4.1.3 Definition

Let G be a graph whose average connectivity is $\overline{k}_G$, diameter is d, adjacency matrix A, order n, and radius r. Then spectral average vertex connectivity $\overline{M}_G$ is given by

$$\overline{M}_G = \begin{cases} \dfrac{\displaystyle\sum_{l=1}^{d}\left(\sum_{u,v}(A^l)_{uv} - \text{trace}(A^l)\right)}{n(n-1)} & , \ d > r \\ \dfrac{\displaystyle\sum_{l=1}^{r+1}\left(\sum_{u,v}(A^l)_{uv} - \text{trace}(A^l)\right)}{n(n-1)} & , \ d = r. \end{cases}$$

4.1.4 Justification of the new result.

In summary, from the work of Beineke et al. (2002), *the average vertex connectivity*, $\overline{k}_G$, has been defined as the quotient of the total number of *internally disjoint paths*, $\sum_{u,v} k_G(u,v)$ and $\binom{n}{2}$. That is
$\overline{k}_G = \dfrac{\sum_{u,v} k_G(u,v)}{\binom{n}{2}}$.
The number of pairwise internally disjoint paths is by counting, which might be complicated in some situations.

This study proposes an estimation of all internally disjoint paths simply by getting the adjacency matrix of the graph. This is given by $M_G = \dfrac{1}{2}\displaystyle\sum_{i=1}^{d}\left(\sum_{u,v}(A^i)_{uv} - \text{trace}(A^i)\right)$ if $d > r$, or $\dfrac{1}{2}\displaystyle\sum_{i=1}^{r+1}\left(\sum_{u,v}(A^i)_{uv} - \text{trace}(A^i)\right)$ if $d = r$. This is so because from the number of walks lemma, $\displaystyle\sum_{i=1}^{l}\sum_{u,v}(A^i)_{uv}$ is twice the number of $u - v$ **walks** from length 1 to length l since the adjacency matrix is a symmetric matrix. The number of closed walks is estimated by $\displaystyle\sum_{i=1}^{l}\text{trace}(A^i)$. Subtracting $\displaystyle\sum_{i=1}^{l}\text{trace}(A^i)$ from $\displaystyle\sum_{i=1}^{l}\sum_{u,v}(A^i)_{uv}$ gives twice the number of $u - v$ **paths**. Hence, M_G estimates the number of **paths** from length 1 to l. In our situation, we have, in case (i), $l = d$ and in case (ii), $l = r + 1$.

When we divide this M_G by $\frac{n(n-1)}{2}$, the $2s$ cancel each other out. That is

$$\frac{\sum_{i=1}^{d}\left(\sum_{u,v}(A^i)_{uv} - \text{trace}(A^i)\right)}{2} \times \frac{2}{n(n-1)} = \frac{\sum_{i=1}^{d}\left(\sum_{u,v}(A^i)_{uv} - \text{trace}(A^i)\right)}{n(n-1)} = \overline{M}_G.$$

Depending on particular graph construction, it can be observed that $\overline{k}_G = \overline{M}_G$ or $\overline{k}_G < \overline{M}_G$. Both equality and inequality situations are shown in the following examples.

4.2. RESULTS

To show the supremacy of the Spectral average vertex connectivity over the combinatoric average vertex connectivity, we show it as an upper bound of Beineke et al. (2002) definition. We use the new definition to prove results on bounds.

4.2.1 Spectral average vertex connectivity of a graph as an upper bound of average vertex connectivity of a graph.

We will use some lemmas such as the spectral number of walks lemma (West, 2001) to prove our main result. As a preliminary to this lemma, one must understand the product rule of counting (Rosen, 2019) which states that if a job of size $k = t + p$ can be done in m ways to finish size t, and then n ways to finish size p, then there are mn ways of doing this job; and the sum rule of counting (Rosen, 2019) which states that if a person can do either job A in m times or job B in n times but can not do both at the same time, then there are $m + n$ ways of doing either of the jobs. The lemma is stated as given below.

Lemma 3 *(West, 2001) Spectral number of walks lemma*

Let X be a graph with adjacency matrix A. Then the number of walks from $u-v$ walks of length l is $(A^l)_{uv}$.

proof

Suppose $l=1$, then $(A^1)_{ij} = \begin{cases} 1 & \text{, uv is an edge ,} \\ 0 & \text{, otherwise.} \end{cases}$

But the edge uv is a $u-v$ walk of length 1, hence the lemma holds when $l=1$.

We now assume that $(A^l)_{u_iv_j} = a_{ij}$ is the number of $u_i - v_j$ walks of length l. Let $A = [b_{ij}]$.

Hence, by this assumption, for each $k = 1, 2, 3, ..., n$, a_{ik} is the number of $u_i - v_k$ of length l and b_{kj} is the number of $u_k - v_j$ of length 1 by the same assumption.

Hence, by product rule of counting, $a_{ik}b_{kj}$ is the number of $u_i - v_j$ walks of length $k+1$ for each $k = 1, 2, 3, ...n$.

But $\sum_{k=1}^{n}(A^l)_{ik}A_{kj} = (A^lA)_{ij} = (A^{l+1})_{ij}$. Therefore, the result holds by induction on l.■

Now, the following theorem is the justification of the reason why the definition is really a representation of average vertex connectivity in spectral graph theory. Thus it emphasizes on both its existence and that it is the upper bound of $\overline{k}_G$. In addition, the fact that, in the next section, we are able to use it to prove the result given by Beineke et al. (2002), is on its own the justification that the equation really deserves the name.

Theorem 4.2.1

Let $\overline{M}_G$ be the spectral average vertex connectivity of graph G, and $\overline{k}_G$ be the average vertex connectivity defined by Beineke et al. (2002). Then $\overline{M}_G$ is an upper bound of $\overline{k}_G$.

proof

Case (i): $d > r$

When $d > r$, we choose d to approximate the longest possible length that can be attained by an internally disjoint path. This estimation is better because every connected graph has a spanning tree and, in a tree, d is the longest possible path.

We would like to count the number of paths of length at most d. If A is the adjacency matrix of graph G then $(A^i)_{uv}$ is the number of walks of length i from u to v.

Hence, the total number of walks of length at most d is $\dfrac{\sum_{i=1}^{d}\sum_{u,v}(A^i)_{uv}}{2}$.

The number of closed walks of length at most d is $\sum_{i=1}^{d} trace(A^i)$.

It follows that the number of paths of length at most d is

$\mathbf{M}_G = \dfrac{\sum_{i=1}^{d}\sum_{u,v}(A^i)_{uv} - \sum_{i=1}^{d} trace(A^i)}{2}$. We divide by 2 because A is symmetric matrix.

But $\sum_{u,v} k_G(u,v)$ is the number of unique paths in graph G; hence

$\sum_{u,v} k_G(u,v) \leqslant \mathbf{M}_G$. Therefore, $\dfrac{\sum_{u,v} k_G(u,v)}{\binom{n}{2}} \leqslant \dfrac{\mathbf{M}_G}{\binom{n}{2}}$.

Let $\dfrac{\sum_{u,v} k_G(u,v)}{\binom{n}{2}} = \overline{k}_G$ and $\dfrac{\mathbf{M}_G}{\binom{n}{2}} = \overline{\mathbf{M}}_G$.

Then $\overline{k}_G \leqslant \overline{\mathbf{M}}_G$.

Hence $\overline{\mathbf{M}}_G = \dfrac{\mathbf{M}_G}{\binom{n}{2}} = \dfrac{\sum_{i=1}^{d}\sum_{u,v}(A^i)_{uv} - \sum_{i=1}^{d} trace(A^i)}{n(n-1)}$ [since $\binom{n}{2} = \dfrac{n(n-1)}{2}$] ■

Case (ii): $d = r$

When $d = r$, we choose $d + 1 = r + 1$ to approximate the longest length that can be attained by an internally disjoint path. We add 1 to take care of the cycles that characterize many graphs in which $d = r$ (Chung, 1988).

We would like to count the number of paths of length at most $r + 1$. If A is the adjacency matrix of graph G then $(A^i)_{uv}$ is the number of walks of length i from u to v.

Hence, the total number of walks of length at most $r+1$ is $\dfrac{\sum_{i=1}^{r+1}\sum_{u,v}(A^i)_{uv}}{2}$.

The number of closed walks of length at most $r+1$ is $\sum_{i=1}^{r+1}\text{trace}(A^i)$.

It follows that the number of paths of length at most d is

$\mathbf{M}_G = \dfrac{\sum_{i=1}^{r+1}\sum_{u,v}(A^i)_{uv} - \sum_{i=1}^{r+1}\text{trace}(A^i)}{2}$.

Since $\sum_{u,v} k_G(u,v)$ is the number of unique paths in graph G, hence

$\sum_{u,v} k_G(u,v) \leqslant \mathbf{M}_G$. Therefore, $\dfrac{\sum_{u,v} k_G(u,v)}{\binom{n}{2}} \leqslant \dfrac{M_G}{\binom{n}{2}}$.

Let $\dfrac{\sum_{u,v} k_G(u,v)}{\binom{n}{2}} = \overline{k}_G$ and $\dfrac{\mathbf{M}_G}{\binom{n}{2}} = \overline{M}_G$. Then $\overline{k}_G \leqslant \overline{\mathbf{M}}_G$.

Hence $\overline{\mathbf{M}}_G = \dfrac{\mathbf{M}_G}{\binom{n}{2}} = \dfrac{\sum_{i=1}^{r+1}\sum_{u,v}(A^i)_{uv} - \sum_{i=1}^{r+1}\text{trace}(A^i)}{n(n-1)}$ [since $\binom{n}{2} = \frac{n(n-1)}{2}$]

■

4.2.2 Upper bounds on the Spectral Average Vertex Connectivity of a graph and Applications.

We want to show that the parameter, $\overline{M}_G$, gives the absolute bounds in its application in complete graphs, thus more tighter than $\overline{k}_G$. It is a verification of the significance of $\overline{M}_G$, as an efficient global measure of connectivity. But before we prove this main result, shown in Theorem 4.2.2, we present several concepts in Lemmas 4, 5 and 6 to clarify some facts asserted in the study. We further generate the upper bounds of $\overline{M}_G$. Later, to show the applicability of this new parameter, we express it and its upper bounds in terms of tree properties and of course, real life applications of the same are presented.

In the first place, we understand that if A is an adjacency matrix of a graph G, then $\sum_{1\le i\le n}(A)_{ij} = v_j$, where v_j is the valence of vertex j (Sheffer, 2003). This is a direct consequence of number of walks lemma. Similarly, $(A)_{ij} = (A)_{ji}$, because any adjacency matrix is a symmetric matrix. This fact leads to the following lemma.

Lemma 4 *(Godsil & Royle, 2001) Adjacency and degree sequence*

If A is an adjacency matrix of a graph G, then $\sum_{1\le i,j\le n} A^2_{ij} = \sum_{k=1}^{n} v_k^2$, where v_k is the valence of vertex k.

proof

From the definition of entry of matrix A^2, we must have $A^2_{ij} = \sum_{k=1}^{n}(A)_{ik}(A)_{kj}$. Hence, we have

$$\sum_{1\le i\le n} A^2_{ij} = \sum_{1\le i\le n}\sum_{k=1}^{n}(A)_{ik}(A)_{kj} = \sum_{k=1}^{n}\sum_{1\le i\le n}(A)_{ik}(A)_{kj} = \sum_{k=1}^{n}(A)_{kj}\sum_{1\le i\le n}(A)_{ik} = \sum_{k=1}^{n}(A)_{kj}v_k.$$

Hence, $\sum_{1\le i,j\le n} A^2_{ij} = \sum_{1\le j\le n}\sum_{k=1}^{n}(A)_{kj}v_k = \sum_{k=1}^{n} v_k \sum_{1\le j\le n}(A)_{kj} = \sum_{k=1}^{n} v_k \sum_{1\le j\le n}(A)_{jk} = \sum_{k=1}^{n} v_k^2.$ ■

In Linear Algebra, if λ is the eigenvalue of A corresponding to eigenvector $\overline{v}$, then λ^2 is the eigenvalue of A^2 corresponding to eigenvector $\overline{v}$. This is so because $A^2\overline{v} = A(A\overline{v}) = A(\lambda\overline{v}) = \lambda^2\overline{v}$. We use this idea together with the lemma below in order to prove the other main result of this study that follow.

Lemma 5 *(Thogersen, 2006) Trace is the sum of eigenvalues*

proof

Suppose A is diagonalizable, then there exists invertible matrix C such that

$C^{-1}AC = D$ is a diagonal matrix, which also implies $CDC^{-1} = A$.

Now, $trace(CDC^{-1}) = trace(A)$. And by the circulatory property of trace, we have $trace(CDC^{-1}) = trace(DCC^{-1}) = \sum_{i=1}^{n}(D)_{ii} = \sum_{i=1}^{n}\lambda_i$ ■

Lemma 6 *(Gross et al., 2019) Spectrum of Complete Graph*

If A is an adjacency matrix of a complete graph of order n, then

$\text{spectrum}(A) = \{(n-1)^1, -1^{n-1}\}$.

proof

Let $y = \langle 1, 1, \cdots, 1\rangle$, then $Ay = (n-1)y$, and if $y^{\perp}$ be a non-zero vector orthogonal to y, then $Ay^{\perp} = -1y^{\perp}$. Thus -1 and $n-1$ are eigenvalues of A. The nullity of $A - -1I = A + I$, where I is identity matrix for matrix multiplication, is $n-1$. Hence, the multiplicity of -1 is $n-1$ and the multiplicity of $n-1$ is $n-(n-1) = 1$. ■

From Lemma 6, we can conclude that $\text{spectrum}(A^2) = \begin{pmatrix} 1 & n-1 \\ (n-1)^2 & (-1)^2 \end{pmatrix}$.

We also use this lemma to prove the significant result of this paper, showing validity of $\overline{M}_G$, and consequently proving the result asserted by Beineke et al. (2002), which is presented in the theorem below.

Theorem 4.2.2 *$\overline{M}_G$ as a tighter upper bound of $\overline{K}_G$.*

Let G be a complete graph of order n such that $\overline{M}_G$ is its spectral average vertex connectivity and $\overline{k}_G$ is its average vertex connectivity defined by Beineke et al. (2002). If A is an adjacency matrix of graph G, then

1. $\overline{M}_G = n-1$

2. $\overline{K}_G \leq \overline{M}_G = n-1$ Beineke et al. (2002) result.

proof

1. In a complete graph, $d = 1 = r$ (West, 2001). Then we use Case (ii) of the definition of $\overline{M}$ to derive the upper bound, so that we substantiate the accurate tightness of our results as compared to those in combinatoric forms. This gives us $r+1 = 1+1 = 2$. We take $M_G^* = \sum_{i=1}^{2}\sum_{u,v}(A^i)_{u,v} - \sum_{i=1}^{2}\text{trace}(A^i)$

   But $\sum_{i=1}^{2}\sum_{u,v} A_{uv}^i = \sum_{u,v} A_{uv}^1 + \sum_{u,v} A_{uv}^2$.

   Since G is complete, then $A_{uv} = \begin{cases} 1 & , if u \neq v, \\ 0 & , otherwise. \end{cases}$

   Hence, $\sum_{u,v} A_{uv}^1 = n(n-1)$.

   If A is adjacency matrix, then $\sum_{u,v} A_{uv}^2 = \sum_{t=1}^{n} v_t^2$, where v_t is the degree/valency of vertex t in the graph.

   But $v_t = n-1$(Complete graph). Therefore $\sum_{u,v} A_{uv}^2 = \sum_{t=1}^{n}(n-1)^2 = n(n-1)^2$.

   And $\sum_{i=1}^{2}\sum_{u,v} A_{uv}^i = n(n-1)+n(n-1)^2 = n(n-1)[1+n-1] = n(n-1)n = (n-1)n^2$.

   Now, $\sum_{i=1}^{2}\text{trace}(A^i) = \text{trace}(A^1) + \text{trace}(A^2) = 0 + \text{trace}(A^2) = \text{trace}(A^2)$.

   But $spectrum(A) = \{(n-1)^1, -1^{n-1}\}$.

   Therefore, $\text{trace}(A^2) = m_1\lambda_1^2 + m_2\lambda_2^2 + ... + m_b\lambda_b^2$, where b is the number of unique eigenvalues, and λ_i are eigenvalues of A.

   So, $\text{trace}(A^2) = m_1\lambda_2 + m_2\lambda_2^2$.

   But $m_1 = 1, m_2 = n-1$ and $\lambda_1 = n-1$, $\lambda_2 = -1$.

Therefore $\text{trace}(A^2) = 1(n-1)^2 + (n-1)(-1)^2 = (n-1)^2 + (n-1) = (n-1)(n-1+1) = n(n-1)$.

This implies that $M_G^* = n^2(n-1) - n(n-1) = n(n-1)(n-1)$.

Hence, $\overline{M}_G = \dfrac{M_G^*}{n(n-1)} = \dfrac{n(n-1)(n-1)}{n(n-1)} = n-1 = k$. ■

2. Thus $\overline{k}_G \leq \overline{M}_G = n-1$. This follows from the preceding proof and the justifying theorem, thereby showing that $\overline{M}_G$ calculates tighter bounds. ■

4.2.3 Upper Bounds of $\overline{M}_G$.

We also derive the tighter upper bound of spectral average connectivity in terms of other parameters like degree sequence, as asserted in the following theorem.

Theorem 4.2.3

Let G be a a connected graph of order n with vertex degree sequence $v_1 > v_2 > ... > v_n$. Then

1. $\overline{M}_G \leq \dfrac{\sum_{i=1}^{d}\sum_{t=1}^{n}(v_t^i - \lambda_t^i)}{n(n-1)}$, when $d > r$

2. $\overline{M}_G \leq \dfrac{\sum_{i=1}^{r+1}\sum_{t=1}^{n}(v_t^i - \lambda_t^i)}{n(n-1)}$, when $d = r$

   where λ_t are eigenvalues of adjacency matrix A of graph G.

Proof

1. Let $S = A + A^2 + ... + A^v$.

   Then $\sum_{i=1}^{d}\sum_{u,v} A^i_{uv} = \sum_{i,j} S_{ij}$ and $\sum_{u,v} A^i \leq \sum_{t=1} v^i_t$.

   Hence $\sum_{i=1}^{d}\sum_{u,v} A^i_{uv} \leq \sum_{i=1}^{d}\sum_{t=1} v^i_t$.

   Therefore $\sum_{i=1}^{d}\sum_{u,v}(A^i)_{uv} - \sum_{i=1}^{d}\text{trace}(A^i) \leq \sum_{i=1}^{d}\sum_{t=1}^{n} v^i_t - \sum_{i=1}^{d}\text{trace}(A^i)$.

   But $\text{trace}(A^i) = \sum_{t=1}^{n}\lambda^i_t$ for each $\lambda_t, t = 1, 2, ..., n$ eigenvalue of A.

   Hence, $M_G \leq \sum_{i=1}^{d}\sum_{t=1}^{n} v^i_t - \sum_{i=1}^{d}\sum_{t=1}^{n}\lambda^i_t$ or $M_G \leq \sum_{i=1}^{d}\sum_{t=1}^{n}(v^i_t - \lambda^i_t)$. This implies that $\overline{M}_G \leq \frac{\sum_{i=1}^{d}\sum_{t=1}^{n}(v^i_t - \lambda^i_t)}{n(n-1)}$.

   Furthermore, since $\overline{k}_G \leq \overline{M}_G$, therefore $\overline{k}_G \leq \overline{M}_G \leq \frac{\sum_{i=1}^{d}\sum_{t=1}^{n}(v^i_t - \lambda^i_t)}{n(n-1)}$.

   Hence, $\overline{k}_G \leq \frac{\sum_{1\leq i\leq d, t\leq n}(v^i_t - \lambda^i_t)}{n(n-1)}$.

2. Similarly for $d = r$, we must have $\overline{k}_G \leq \frac{\sum_{1\leq i\leq r+1, 1\leq t\leq n}(v^i_t - \lambda^i_t)}{n(n-1)}$ ■

We also prove the the corollary of Theorem 4.2.3 to show that the results are consistent, as below.

Corollary 5

Let Δ be the maximum degree. If $\Delta = v_1$ where $v_1 \geq v_2 \geq v_3 \geq ... \geq v_n$ are valences of vertices of graph G, then

1. $\overline{M}_G \leq \frac{\sum_{1\leq i\leq d} n\Delta^i - \sum_{1\leq i\leq d, 1\leq t\leq n}\lambda^i_t}{n(n-1)}$, if $d > r$ and

2. $\overline{M}_G \leq \frac{\sum_{1\leq i\leq r+1} n\Delta^i - \sum_{1\leq i\leq r+1, 1\leq t\leq n}\lambda^i_t}{n(n-1)}$, if $d = r$.

Proof

Since $\Delta = v_1 \geq v_t$, for all $t = 2, 3, ..., n$. It follows that $v_t \leq \Delta$, for all $t = 1, 2, 3, ..., n$. This implies that $v_t^i \leq \Delta^i$, since $v_t, \Delta \in \mathbb{N}$ and $\sum_{i=1}^{n} v_t^i \leq \sum_{t=1}^{n} \Delta^i = n\Delta^i$.

Hence, we have

$$
\begin{aligned}
M_G^{**} &= \sum_{i=1}^{d} \sum_{t=1}^{n} (v_t^i - \lambda_t^i) \\
&= \sum_{i=1}^{d} \sum_{t=1}^{n} v_t^i - \sum_{i=1}^{d} \sum_{t=1}^{n} \lambda_t^i \\
&= \sum_{i=1} n\Delta^i - \sum_{i \leq d, t \leq n} \lambda_t^i
\end{aligned}
$$

$$
\therefore \overline{M}_G = \frac{M_G^{**}}{n(n-1)} = \frac{\sum_{1 \leq i \leq d} n\Delta^i - \sum_{1 \leq i \leq d, 1 \leq t \leq n} \lambda_t^i}{n(n-1)} \blacksquare
$$

And the other results follow.

The validity of the results derived in Theorem 4.2.3 and Corollary 5 is checked in a complete graph, and

this explains as to why the Spectral average vertex Connectivity is more efficient than the Combinatoric average vertex connectivity.

4.2.4 Spectral Definition in a Complete Graph.

Since in complete graph

$$
\begin{aligned}
d =& 1 = r \\
\Delta =& n - 1 \\
\lambda_1 =& n - 1, \text{ with multiplicity } (1) \\
\lambda_2 =& -1, \text{ with multiplicity } (n-1).
\end{aligned}
$$

$$
\begin{aligned}
Then \sum_{i \le r+1} n\Delta^i =& \sum_{i \le 2} n(n-1)^i \\
=& n(n-1)^1 + n(n-1)^2 \\
=& n(n-1)(1 + n - 1) \\
=& n^2(n-1)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i \le r+1, t \le n} \lambda_t^i =& \sum_{i \le r+1} (1(n-1)^i) + (-1)^i(n-1)) \\
=& 0 + 1(n-1)^2 + (n-1)(-1)^2 \\
=& (n-1)^2 + (n-1) \\
=& (n-1)(n-1+1) \\
=& n(n-1).
\end{aligned}
$$

So,

$$\begin{aligned}\overline{M}_G \leq & \frac{n^2(n-1)-n(n-1)}{n(n-1)} \\ = & \frac{n(n-1)(n-1)}{n(n-1)} \\ = & n-1 \blacksquare\end{aligned}$$

This proves Beineke et al. (2002) result successfully (substantiating the tightness of the bounds). It shows that the results on Theorems 4.2.3 and Corollary 5 are even more consistent with properties of average vertex connectivity.

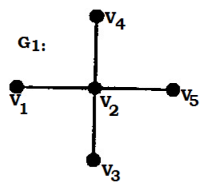
4.2.5 Examples to demonstrate that the two definitions are the same.

Examples where the Equality holds ($\overline{k}_G = \overline{M}_G$).

We will take examples from Beineke et al. (2002) paper and compare results.

Example 1

From Figure 1, page 32, G_1 (Beineke et al., 2002) - Calculating $\overline{k}_G$ of G_1.



$$\overline{k}_G = \frac{\sum_{u,v} k_G(u,v)}{(n(n-1))/2} = \frac{10}{(5(5-1))/2} = \frac{10}{10} = 1$$

This study - calculating $\overline{M}_G$ of G_1

Table 4.1: Matrix for internally disjoint paths of G_1

	v_1	v_2	v_3	v_4	v_5
v_1	0	1	1	1	1
v_2		0	1	1	1
v_3			0	1	1
v_4				0	1
v_5					0

We want to find the average connectivity of the same graph using $\overline{M}_G$.

We begin by identifying the radius(r) and diameter(d) of this graph from the distance matrix given below.

Table 4.2: Distance Matrix of G_1

	v_1	v_2	v_3	v_4	v_5	Eccentricity
v_1	0	1	2	2	2	2
v_2	1	0	1	1	1	1
v_3	2	1	0	2	2	2
v_4	2	1	2	0	2	2
v_5	2	1	2	2	0	2

From the table, $r = 1$ and $d = 2$; which means $d > r$. So we use case 1. That's

$$\overline{M}_G = \frac{\sum_{i=1}^{d} \left(\sum_{u,v} (A^i)_{uv} - \text{trace}(A^i) \right)}{n(n-1)}, \; d > r$$

Substituting, we have $\dfrac{\sum A^1_{uv} + \sum A^2_{uv} - \text{trace(A)} - \text{trace}(A^2)}{5(5-1)}$

Getting the adjacency matrix(A) of the graph, we employ MATLAB with the following algorithm:

```
function experiment_Malota1

A=[0 1 0  0 0;...

   1 0 1  1  1;...

   0 1  0  0 0;...

   0  1  0  0 0;...

   0  1  0  0 0];
```

```
b=A^2;

s=sum(sum(A))+ sum(sum(b));

t=trace(A)+trace(b);

[n p]=size(A);

n=n;

N=s-t;

d=n*(n-1);

M_bar=N/d     end
```

The output is as follows:

```
>> experiment_Malota1

A =  0     1     0     0     0

     1     0     1     1     1

     0     1     0     0     0

     0     1     0     0     0

     0     1     0     0     0

b =  1     0     1     1     1

     0     4     0     0     0

     1     0     1     1     1

     1     0     1     1     1

     1     0     1     1     1

s = 28

t = 8

n = 5

p = 5
```

```
n = 5

N = 20

d = 20

M_bar = 1
```

Hence, $\overline{k}_G = \overline{M}_G$ ■

Example 2

In Figure 1, page 32, G_2 (Beineke et al., 2002) - Calculating $\overline{k}_G$ of G_2.

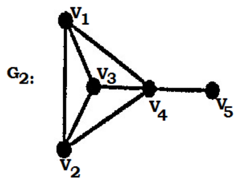


Table 4.3: Matrix for internally disjoint paths of G_2

	v_1	v_2	v_3	v_4	v_5
v_1	0	3	3	3	1
v_2		0	3	3	1
v_3			0	3	1
v_4				0	1
v_5					0

$$\overline{k}_G = \frac{\sum_{u,v} k_G(u,v)}{(n(n-1))/2} = \frac{22}{(5(5-1))/2} = \frac{22}{10} = 2.2$$

This study - calculating $\overline{M}_G$ of G_2.

We want to find the average connectivity of the same graph using $\overline{M}_G$. We identify the radius(r) and diameter(d) of this graph from the distance matrix given below.

Table 4.4: Distance Matrix of G_2

	v_1	v_2	v_3	v_4	v_5	Eccentricity
v_1	0	1	1	1	2	2
v_2	1	0	1	1	2	2
v_3	1	1	0	1	2	2
v_4	1	1	1	0	1	1
v_5	2	2	2	1	0	2

From the table, $r = 1$ and $d = 2$; which means $d > r$. So we use case 1. That's

$$\overline{M}_G = \frac{\sum_{i=1}^{d}\left(\sum_{u,v}(A^i)_{uv} - \text{trace}(A^i)\right)}{n(n-1)}, \; d > r.$$

Substituting, we have $\frac{\sum A^1_{uv} + \sum A^2_{uv} - \text{trace(A)} - \text{trace}(A^2)}{5(5-1)}$.

Getting the adjacency matrix(A) of the graph, we employ MATLAB with the following algorithm:

```
function experiment_Malota2

A=[0 1 1 1 0;...

   1 0 1  1  0;...

   1 1  0  1 0;...

   1  1  1  0 1;...

   0  0  0  1 0]

b=A^2

s=sum(sum(A))+ sum(sum(b))

t=trace(A)+trace(b)

[n p]=size(A)

n=n

N=s-t

d=n*(n-1)

M_bar=N/d
```

```
end
```

The output is as follows:

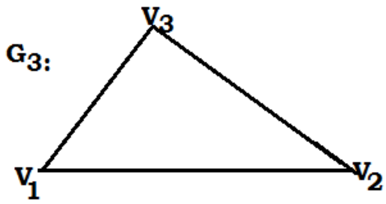
```
>> experiment_Malota2

A =  0     1     1     1     0

     1     0     1     1     0

     1     1     0     1     0

     1     1     1     0     1

     0     0     0     1     0

b =  3     2     2     2     1

     2     3     2     2     1

     2     2     3     2     1

     2     2     2     4     0

     1     1     1     0     1

s = 58

t = 14

n = 5

p = 5

n = 5

N = 44

d = 20

M_bar = 2.2000
```

Hence, $\overline{k}_G = \overline{M}_G$ ■

Example 3

This example is deliberately chosen so that case 2 of the definition of $\overline{M}_G$ should be used.

Using a regular graph with n=3 (k_3), we label it G_3.



Calculating $\overline{k}_G$ of G_3.

Table 4.5: Matrix for internally disjoint paths of G_3

	v_1	v_2	v_3
v_1	0	2	2
v_2		0	2
v_3			0

$$\overline{k}_G = \frac{\sum_{u,v} k_G(u,v)}{(n(n-1))/2} = \frac{6}{(3(3-1))/2} = \frac{6}{3} = 2$$

This study - calculating $\overline{M}_G$ of G_3.

We want to find the average connectivity of the same graph using $\overline{M}_G$.

We begin by identifying the radius(r) and diameter(d) of this graph from the distance matrix given below.

Table 4.6: Distance Matrix of G_3

	v_1	v_2	v_3	Eccentricity
v_1	0	1	1	1
v_2	1	0	1	1
v_3	1	1	0	1

From here, $r = d = 1$. So we use case 2. That is

$$\overline{M}_G = \frac{\sum_{i=1}^{r+1}\left(\sum_{u,v}(A^i)_{uv} - \text{trace}(A^i)\right)}{n(n-1)},\ d = r.$$

We add up to $r = 1 + 1 = 2$

Substituting, we have $\dfrac{\sum A^1_{uv} + \sum A^2_{uv} - \text{trace(A)} - \text{trace}(A^2)}{5(5-1)}$

Getting the adjacency matrix(A) of the graph, we employ MATLAB with the following algorithm:

```
function experiment_Rid1

A= [0  1  1;...

    1  0  1;...

    1  1  0]

p=A^2

s=sum(sum(A))+sum(sum(p))

t=trace(A)+trace(p)

[n p]=size(A)

n=n

N=s-t

d=n*(n-1)

M_bar=N/d

end
```

The output is as follows:

```
>> experiment_Rid1

A =   0      1      1

      1      0      1
```

```
      1     1     0

p =   2     1     1

      1     2     1

      1     1     2

s = 18

t = 6

n = 3

p = 3

n = 3

N = 12

d = 6

M_bar = 2
```

Hence, $\overline{k}_G \leq \overline{M}_G$ ■

This example verifies the result of Beineke et al. (2002) which states that for any complete graph $\overline{k}_G \leq n-1$ where n is order of a graph. This example also verifies the fact that for a complete graph $\overline{M}_G = n-1$, which is contribution of this study.

Example showing $\overline{k}_G < \overline{M}_G$.

Depending on the a particular graph construction, the value of $\overline{k}_G$ might be smaller than the value of $\overline{M}_G$.

Example 4

Using graph H_1 in Beineke's et al. (2002) paper, page 33, Figure 2.

Calculating $\overline{k}_G$ of H_1.

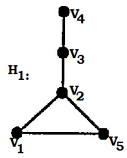


Table 4.7: Matrix for internally disjoint paths of H_1

	v_1	v_2	v_3	v_4	v_5
v_1	0	2	1	1	2
v_2		0	1	1	2
v_3			0	1	1
v_4				0	1
v_5					0

$$\overline{k}_G = \frac{\sum_{u,v} k_G(u,v)}{(n(n-1))/2} = \frac{13}{(5(5-1))/2} = \frac{13}{10} = 1.3$$

This study - calculating $\overline{M}_G$ of H_1.

We want to find the average connectivity of the same graph using $\overline{M}_G$.

We begin by identifying the radius(r) and diameter(d) of this graph from the distance matrix given below.

Table 4.8: Distance Matrix of H_1

	v_1	v_2	v_3	v_4	v_5	Eccentricity
v_1	0	1	2	3	1	3
v_2	1	0	1	2	1	2
v_3	2	1	0	1	2	2
v_4	3	2	1	0	3	3
v_5	1	1	2	3	0	3

From here, $r = 2$ and $d = 3$; which means $d > r$. So we use case 1. That's

$$\overline{M}_G = \frac{\sum_{i=1}^{d}\left(\sum_{u,v}(A^i)_{uv} - \text{trace}(A^i)\right)}{n(n-1)}, \ d > r$$

Substituting, we have $\dfrac{\sum A^1_{uv} + \sum A^2_{uv} + \sum A^3_{uv} - \ \text{trace}(A) - \text{trace}(A^2) - \text{trace}(A^3)}{5(5-1)}$

Getting the adjacency matrix(A) of the graph, we employ MATLAB with the following algorithm:

```
function experiment_Rid2
A=[0  1  0  0 1;...
   1  0  1  0 1;...
   0  1  0  1 0;...
   0  0  1  0 0;...
   1  1  0  0 0]
b=A^2
b1=A^3
s=sum(sum(A))+ sum(sum(b))+sum(sum(b1))
t=trace(A)+trace(b)+trace(b1)
[n p]=size(A)
n=n
N=s-t
d=n*(n-1)
M_bar=N/d            end
```

The output is as follows:

```
>> experiment_Rid2
A =  0     1     0     0     1
     1     0     1     0     1
     0     1     0     1     0
     0     0     1     0     0
```

```
     1     1     0     0     0

b =  2     1     1     0     1

     1     3     0     1     1

     1     0     2     0     1

     0     1     0     1     0

     1     1     1     0     2

b1 = 2     4     1     1     3

     4     2     4     0     4

     1     4     0     2     1

     1     0     2     0     1

     3     4     1     1     2

s =  80

t =  16

n =  5

p =  5

n =  5

N =  64

d =  20

M_bar = 3.2000
```

Hence, $\overline{k}_G < \overline{M}_G$ ■

The above presented examples show that, really, $\overline{k}_G \leq \overline{M}_G$.

4.2.6 Application in Trees.

To emphasize its effectiveness, the spectral definition of average vertex connectivity and its bounds derived above are here expressed in terms of properties of a tree.

To begin with, we take into consideration the following facts about trees:

(i) $\sum_{t=1}^{n} \lambda_t^2 = 2|E|$, where $|E|$ is the size of graph, say G (number of edges).

(ii) If G is a tree, then it is a bipartite graph, hence $\sum_{t}^{b} \lambda_t^{2i+1} = 0, \forall i \in \mathbf{N}$. However, in a tree $|E| = n - 1$ (West, 2001). So that $\sum_{t=1}^{n} \lambda_t^2 = \sum_{t=1}^{n} v_t = 2|E| = 2(n-1)$.

Hence, there exists $\omega \in \mathbb{R}$ such that

$$\begin{aligned}\sum_{i=1}^{n} \lambda_t^i =& \omega \sum_{t=1}^{n} \lambda_i^2 \\ =& \omega \sum_{t=1}^{n} v_t \\ =& 2\omega(n-1).\end{aligned}$$

We choose $\omega \geq \lfloor \frac{d}{2} \rfloor$, if $d > r$ and $\omega \geq \lfloor \frac{r+1}{2} \rfloor$, if $d = r$.

Hence, from Corollary 5 we get the following corollary.

Corollary 6

Let $\overline{M}_G$ be the average vertex connectivity of graph G of order n, whose maximum degree, diameter and radius are denoted by Δ, d and r respectively. If G is a tree, then

$$
\overline{M}_G = \begin{cases} \dfrac{\displaystyle\sum_{i=1}^{d} n\Delta^i - 2\left\lfloor \frac{d}{2} \right\rfloor (n-1)}{n(n-1)} & , \ d > r \\ \dfrac{\displaystyle\sum_{i=1}^{r+1} n\Delta^i - 2\left\lfloor \frac{r+1}{2} \right\rfloor (n-1)}{n(n-1)} & , \ d = r. \end{cases}
$$

The validity of the results in Corollary 6 are checked in the examples to follow.

Example 5

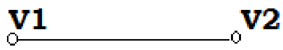


Figure 4.1: First Tree

Table 4.9: First Tree Distance Matrix

	v_1	v_2	$e(v_i)$
v_1	0	1	1
v_2	1	0	1

From the graph in Figure 1, $n = 2$ and $\Delta = 1$.

From the distance matrix above, $d = 1 = r$.

$$
\begin{aligned}
\overline{M}_G &= \frac{\displaystyle\sum_{i=1}^{2} 2(1)^i - 2\left\lfloor \frac{2}{2} \right\rfloor (2-1)}{2(1)} \\
&= \frac{4-2}{2} \\
&= 1 = k
\end{aligned}
$$

Remarks

(i)This is an example where $\overline{k}_G = \overline{M}_G$

(ii) If we delete one vertex, we discover that it is only one vertex that is affected.

The expected number of vertices that are affected is approximated by

$\lfloor \overline{M}_G \rfloor = \lfloor 1 \rfloor = 1$. This is also the valence of the vertex that can be deleted to lead to a trivial graph, showing that $\overline{M}_G$ is applicable in trees.

Example 6

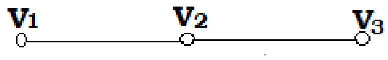


Figure 4.2: Second Tree

Table 4.10: Second Tree Distance Matrix

	v_1	v_2	v_3	$e(v_i)$
v_1	0	1	2	2
v_2	1	0	1	1
v_3	2	1	0	2

$$\therefore \overline{M}_G = \frac{\sum_{i=1}^{2}(3)(2)^i - 2\lfloor \frac{2}{2} \rfloor (3-1)}{3(3-1)}$$
$$= \frac{(3)(2) + (3)(2)^2 - 2(1)(2)}{6}$$
$$= \frac{6 + 12 - 4}{6}$$
$$= \frac{14}{6}$$
$$=2.3$$

remarks

(i) This is an example where $\overline{k}_G < \overline{M}_G$

(ii) If we delete one vertex, the whole graph will be disconnected, that is $k = 1 = \overline{M}_G$.

However, there are 2 vertices which are affected.

The number of vertices affected by deleting these vertices is predicted by $\lfloor \overline{M}_G \rfloor = \lfloor 2.3 \rfloor = 2$. This is also the valence of a vertex that can be deleted to produce the effect. It is also the number of vertices that can be deleted to produce a trivial graph.

Example 7

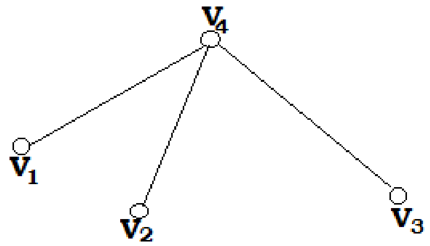


Figure 4.3: Third Tree

Table 4.11: Third Tree Distance Matrix

	v_1	v_2	v_3	v_4	$e(v_i)$
v_1	0	2	2	1	2
v_2	2	0	2	1	2
v_3	2	2	0	1	2
v_4	1	1	1	0	1

From the graph,$n = 4$ and $\Delta = 3$

From the matrix, $d = 2 > 1 = r$

$$\therefore \overline{M}_G = \frac{\sum_{i=1}^{2} 4(3)^i - 2\lfloor\frac{2}{2}\rfloor(4-1)}{4(4-1)}$$

$$= \frac{4(3) + 4(3)^2 - 2(1)(3)}{4(3)}$$

$$= \frac{42}{12}$$

$$=3.3$$

Remarks

(i) This is an example where $\overline{k}_G \leq \overline{M}_G$

(ii) If you delete one vertex, the whole graph will be disconnected; so $k = 1$. Three vertices will be affected, which is $3 = \lfloor 3.5 \rfloor$. This is also the valence of a vertex that can be deleted to produce the effect. It is also the number of vertices that can be deleted to produce a trivial graph.

CHAPTER 5

CONCLUSION

The study proposed a new definition of average vertex connectivity in spectral form, using properties of adjacency matrix. This has been derived using several lemmas and theorems, and the results verified are more consistent with average connectivity. The proofs of the theorems indicate that this new measure of connectivity is obviously easy to apply and calculate. It is also shown that the spectral average connectivity is the upper bound of average connectivity defined by Beineke et al. (2002), hence the upper bound for connectivity. It has many advantages over that of Beineke et al. (2002) such that it is reliably used to prove several results in graph theory, such as that of Beineke et al. (2002); ($\overline{k}_G \leq n - 1$).

Several lemmas preceding theorems are used to justify that the spectral definition of average connectivity is really a reliable global measure of connectivity. Examples are done in different families of graphs such as complete graphs, trees, to show the validity and hence the usability of this parameter.

Just as Beineke et al. (2002) reinforced the average vertex connectivity, $\overline{k}_G$, with bounds in combinatoric forms, this study has developed several upper bounds for spectral definition of average vertex connectivity, $\overline{M}_G$, that can be verified very more easily, making it much more attractive for applications. Cases and examples for verification are provided, using various graphs. Validity of the bounds in spectral forms is done by comparing the tightness of spectral and combinatorial upper bounds of average connectivity. Evidence is exhibited in the proofs that the bounds in spectral forms are even tighter and easy to compute than those in combinatoric forms, making it spectacular over many global measures of connectivity.

Extension is made where the spectral definition of average connectivity is expressed in terms of tree properties, with given relevant examples, showing the effects of deleting one vertex to disconnect the graph, and use of the spectral average connectivity to approximate the number of vertices affected after deleting one. This makes the parameter more applicable in graph theory and to real life situations.

While the constructions and calculations in this source are accurately done, they by no means exhaust the possibilities. Production of Distance Matrix to identify eccentricities remains a non-simplified task.

Further work can be done to combine eccentricity directly in the definition formula to make adjacency matrix exclusively paramount in calculating average vertex connectivity of any graph.

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